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Rational Approximation to e^{-x}

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INTRODUCTION

It was observed by Cody *et al.* [2] that e^{-x} can be uniformly approximated on $[0, \infty)$ by reciprocals of polynomials of degree n with the error c^n where for c we may take 0.43501.... Besides, they showed that if π_n denotes the collection of all real polynomials of degree at most n, and

$$\lambda_{0,n} \equiv \inf_{p_n \in \pi_n} \left\{ \sup_{0 \leqslant x < \infty} \left| e^{-x} - \frac{1}{p_n(x)} \right| \right\},$$

then

$$\overline{\lim_{n\to\infty}} (\lambda_{0,n})^{1/n} \geqslant \tfrac{1}{6}.$$

Later, it was proved by Schönhage [5] that

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n} = \tfrac{1}{3}.$$

Newman [4] investigated whether one could achieve better than a c^n error by using general rational functions, and showed that one cannot. He in fact proved the following

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THEOREM A. *Let $P(x)$, $Q(x)$ be any polynomials of degree $<n$. There must be a point on the positive axis where*

$$\left| e^{-x} - \frac{P(x)}{Q(x)} \right| > 1280^{-n}. \tag{1}$$

It has been commonly believed that the number 1280 appearing in (1) is far from being the best possible. In our attempt to improve upon this number we have been able to prove the following

THEOREM 1. *Let $P(x)$ and $Q(x)$ be any polynomials of degree at most n. Then*

$$\max_{x\geqslant 0} \left| e^{-x} - \frac{P(x)}{Q(x)} \right| > 308^{-n}. \tag{2}$$

Our method of proof not only gives an improvement on Newman's constant but can also be extended to a more general situation. We use it to prove the following result which solves a problem raised by Erdös and Reddy [3, Problem 4].

THEOREM 2. *Let*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_0 > 0, \quad a_k \geqslant 0 \quad (k \geqslant 1)$$

be an entire function of order ρ $(0 < \rho < \infty)$ type τ and lower type ω $(0 < \omega \leqslant \tau < \infty)$. Then there is a constant $C > 1$ such that for any polynomials $P(x)$ and $Q(x)$ of degree at most n we have

$$\max_{x\geqslant 0} \left| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right| \geqslant \frac{1}{C^n}.$$

In fact, if $\mathscr{P}_n$ denotes the class of all polynomials of degree at most n and

$$\lambda_{n,n}(f) = \inf_{P,Q\in\mathscr{P}_n} \left\{ \sup_{x\geqslant 0} \left| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right| \right\}$$

then

$$\varliminf_{n\to\infty} (\lambda_{n,n}(f))^{1/n} \geqslant \frac{1}{(16 + 8(2^{1/2}))^2} \exp(-\rho^{-2} - 2\tau\omega^{-1}).$$

Here we have not even attempted to get a better bound.

LEMMAS

In what follows we shall denote by $T_n(x)$ the nth Chebyshev polynomial

$$T_n(x) = \cos(n \arccos x).$$

First we recall a well-known estimate for polynomials with a given bound on an interval.

LEMMA 1. *Let $P_n(x)$ be a polynomial of degree at most n and $[a, b]$ some interval such that $|P_n(x)| \leqslant 1$ for all $x \in [a, b]$. Then for every $c > b$ we have*

$$|P_n(c)| \leqslant T_n((2c - b - a)/(b - a)).$$

This inequality may become quite crude, if $|P_n(x)|$ actually has some curved majorant on $[a, b]$. A more appropriate estimate in this case (but with $c < a$ instead of $c > b$) is given by the following

LEMMA 2. *Let $P_n(x)$ be a polynomial of degree at most n having a continuous positive majorant $M(x)$ on some interval $[a, b]$, i.e.,*

$$|P_n(x)| \leqslant M(x) \qquad \textit{for all } x \in [a, b].$$

Then for every $c < a$ we have

$$|P_n(c)| \leqslant \frac{1}{r^n} \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2) \log M(((b-a)/2) \cos\theta + (b+a)/2)}{1 + 2r\cos\theta + r^2}\, d\theta\right\}, \tag{3}$$

where

$$r = \frac{b + a - 2c}{b - a} - \left\{\left(\frac{b + a - 2c}{b - a}\right)^2 - 1\right\}^{1/2}.$$

Proof. Put

$$f(z) = z^n P_n\left(\frac{b-a}{2}\left(\frac{z + z^{-1}}{2}\right) + \frac{b+a}{2}\right).$$

Then $f(z)$ is a holomorphic function,

$$f(-r) = (-r)^n P_n(c)$$

and for all $\theta \in [0, 2\pi]$

$$|f(e^{i\theta})| \leqslant M\left(\frac{b-a}{2}\cos\theta + \frac{b+a}{2}\right) \tag{4}$$

It is well known (see, for example, [1, p. 168]) that the Poisson integral $H(z)$, defined by

$$H(\rho e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-\rho^2)\log M(((b-a)/2)\cos\theta + (b+a)/2)}{1 - 2\rho\cos(\theta-\varphi) + \rho^2}\, d\theta,$$

is a harmonic function in the unit disk. Thus according to (4), $H(z)$ is a harmonic majorant of the subharmonic function $\log |f(z)|$, and so

$$\log |f(-r)| = \log\{r^n \mid P_n(c)|\} \leqslant H(re^{i\pi}),$$

which completes the proof of Lemma 2.

In our applications the majorant $M(x)$ will be a Chebyshev polynomial, and the integral appearing on the right-hand side of (3) will be estimated with the help of the following

LEMMA 3. *Let c and d be positive numbers such that $d - c > 1$. Then, for $0 < r < 1$,*

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)\log T_n(c\cos\theta + d)}{1 + 2r\cos\theta + r^2}\, d\theta \leqslant \log T_n(d - rc).$$

Proof. By our assumptions $| cz + d | > 1$ for $| z | \leqslant 1$. Hence $T_n(cz + d)$ does not vanish in the closed unit disk, and consequently,

$$f(z) = \log | T_n(cz + d)|$$

defines there a harmonic function. By the Poisson formula we have

$$\log T_n(d - rc) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)\log | T_n(ce^{i\theta} + d)|}{1 + 2r\cos\theta + r^2}\, d\theta.$$

Since the zeros of $T_n(z)$ are all real

$$| T_n(ce^{i\theta} + d)| \geqslant T_n(c\cos\theta + d)$$

for all $\theta \in [0, 2\pi]$, and hence we have the desired result.

We shall also need the following trivial estimate for the nth Chebyshev polynomial $T_n(x)$.

LEMMA 4. *For all real x such that $| x | > 1$, we have*

$$| T_n(x)| \leqslant \tfrac{1}{2} | 2x |^n.$$

PROOFS OF THE THEOREMS

Proof of Theorem 1. Put $\beta = \log 308 = 5.7300997...$, and let α and γ be two numbers with

$$0 < \alpha < \beta < \gamma$$

which we will choose later.

Assume now that Theorem 1 is false. Then there exist polynomials $P(x)$ and $Q(x)$ of degree at most n such that

$$\left| e^{-x} - \frac{P(x)}{Q(x)} \right| \leqslant e^{-\beta n} \tag{5}$$

throughout the positive real axis. We may normalize $Q(x)$ so that

$$\max_{0 \leqslant x \leqslant \alpha n} | Q(x)| = 1. \tag{6}$$

Then at a point $\xi = n\xi^* \in [0, \alpha n]$ where $| Q(\xi)| = 1$ we obtain

$$| P(\xi)| \geqslant e^{-\xi} - e^{-\beta n}. \tag{7}$$

Let us now estimate $| P(\xi)|$ from above. By (6) and Lemma 1 we get

$$| Q(nt)| \leqslant T_n\left(\frac{2t - \alpha}{\alpha}\right) \qquad \text{for} \quad t \geqslant \alpha.$$

Hence it follows from (5) that for $t \in [\beta, \gamma]$,

$$| P(nt)| \leqslant (e^{-nt} + e^{-\beta n}) \, | Q(nt)| \leqslant 2e^{-\beta n} T_n\left(\frac{2t - \alpha}{\alpha}\right).$$

This inequality gives us a curved majorant of $| P(x)|$ on the interval $[n\beta, n\gamma]$. With the notation

$$w = \frac{\gamma + \beta - 2\xi^*}{\gamma - \beta} \qquad \text{and} \qquad r = w - (w^2 - 1)^{1/2}$$

we obtain by Lemma 2

$$| P(\xi)| \leqslant \frac{1}{r^n} \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2) \log M(\theta)}{1 + 2r\cos\theta + r^2} \, d\theta\right\},$$

where

$$M(\theta) = 2e^{-\beta n} T_n\left(\frac{(\gamma - \beta)\cos\theta + \gamma + \beta - \alpha}{\alpha}\right).$$

Using the estimate given in Lemma 3 it follows that

$$|P(\xi)| \leqslant \frac{1}{r^n} 2e^{-\beta n} T_n \left(\frac{\gamma + \beta - \alpha - r(\gamma - \beta)}{\alpha}\right).$$

Finally, simplifying the right-hand side with the help of Lemma 4, we obtain

$$|P(\xi)| \leqslant \left\{e^{-\beta} 2 \left(\frac{\gamma + \beta - \alpha}{\alpha r} - \frac{\gamma - \beta}{\alpha}\right)\right\}^n. \tag{8}$$

Now we compare the inequalities (7) and (8). If

$$\Omega_n(\xi^*, \alpha, \gamma) = \frac{\alpha r e^{-\xi^*}(1 - e^{-(\beta-\xi^*)n})^{1/n}}{2e^{-\beta}(\gamma + \beta - \alpha - (\gamma - \beta)\, r)},$$

then we must obviously have

$$\Omega_n(\xi^*, \alpha, \gamma) \leqslant 1. \tag{9}$$

Notice that $r = r(\xi^*)$ depends on ξ^*. Differentiating

$$\frac{r(\xi^*)\, e^{-\xi^*}}{\gamma + \beta - \alpha - r(\xi^*)(\gamma - \beta)}$$

with respect to ξ^* we find that this expression is a decreasing function of ξ^* in $[0, \alpha]$ if the inequality

$$2\,\frac{\gamma + \beta - \alpha}{\gamma - \beta} < \frac{1 - \lambda^2}{2\lambda} (\gamma + \beta - \alpha - \lambda(\gamma - \beta)), \tag{10}$$

where

$$\lambda = r(\alpha) = \frac{\gamma + \beta - 2\alpha}{\gamma - \beta} - \left\{\left(\frac{\gamma + \beta - 2\alpha}{\gamma - \beta}\right)^2 - 1\right\}^{1/2}$$

is satisfied. Hence, subject to the condition that (10) holds we have

$$\Omega_n(\xi^*, \alpha, \gamma) \geqslant \frac{\alpha\lambda e^{\beta-\alpha}}{2(\gamma + \beta - \alpha - \lambda(\gamma - \beta))} (1 - e^{(\alpha-\beta)n})^{1/n}.$$

Now we set $\alpha = 6/5$, $\gamma = 30$, and by a numberical calculation we obtain

$$0.4320446 < \lambda < 0.4320447,$$

$$\frac{\alpha\lambda e^{\beta-\alpha}}{2(\gamma + \beta - \alpha - \lambda(\gamma - \beta))} \geqslant 1.00014.$$

Furthermore, we find that (10) is satisfied. Therefore,

$$\Omega_n(\xi^*, 1.2, 30) \geqslant 1.00014(1 - e^{-4n})^{1/n}.$$

This contradicts (9) if $n \geqslant 3$. But for $n = 1, 2$ (actually for $1 \leqslant n \leqslant 14$) calculations given in [2] show that even

$$\max_{x \geqslant 0} \left| e^{-x} - \frac{P(x)}{Q(x)} \right| > 15^{-n}$$

This cimpletes the proof of Theorem 1.

Proof of Theorem 2. Without loss of generality we may assume that $a_0 = 1$. Now suppose that the theorem is false. Then, for a sufficiently small $\epsilon > 0$ and infinitely many n there exist polynomials $P(x)$ and $Q(x)$ of degree at most n for which

$$\sup_{x \geqslant 0} \left| \frac{1}{f((x/(\omega - \epsilon))^{1/\rho})} - \frac{P(x^{1/\rho})}{Q(x^{1/\rho})} \right| < e^{-\beta n}, \tag{11}$$

where

$$\beta = \frac{1}{\rho^2} + 2\,\frac{\tau + \epsilon}{\omega - \epsilon} + 2\log(16 + 8(2^{1/2})).$$

Here we normalize $Q(x)$ so that

$$\max_{0 \leqslant x \leqslant n} | Q(x^{1/\rho})| = 1. \tag{12}$$

Denoting by $\xi = n\xi^* \in [0, n]$ a point where this maximum is attained we find

$$| P(\xi^{1/\rho})| \geqslant \frac{1}{f((n\xi^*/(\omega - \epsilon))^{1/\rho})} - e^{-\beta n}.$$

According to the assumption of our theorem $f(x^{1/\rho})$ may be represented for $x \geqslant 0$ as

$$f(x^{1/\rho}) = e^{x\theta(x)} \tag{13}$$

with

$$\omega = \varliminf_{x \to \infty} \theta(x) \leqslant \varlimsup_{x \to \infty} \theta(x) = \tau. \tag{14}$$

Consequently, for sufficiently large n we have

$$P(\xi^{1/\rho}) \geqslant \exp\left(-\left(\frac{\tau + \epsilon}{\omega - \epsilon}\right) n\right) - e^{-\beta n}. \tag{15}$$

Let us now estimate $|P(\xi^{1/\rho})|$ from above. Considering (12) we obtain by Lemma 1 that

$$|Q((nt)^{1/\rho})| \leqslant T_n(2t^{1/\rho} - 1) \qquad \text{for} \quad t \geqslant 1.$$

Furthermore, by (11) we have

$$|P((nt)^{1/\rho})| < \left\{\frac{1}{f((nt/(\omega - \epsilon))^{1/\rho})} + e^{-\beta n}\right\} |Q((nt)^{1/\rho})|.$$

If n is large enough then, because of (13) and (14),

$$\frac{1}{f((nt/(\omega - \epsilon))^{1/\rho})} < e^{-nt} \qquad \text{for} \quad t \in [\beta, \gamma].$$

Hence taking all these inequalities into account, we get

$$|P((nt)^{1/\rho})| \leqslant 2e^{-\beta n} T_n(2t^{1/\rho} - 1) \qquad \text{for} \quad t \in [\beta, \gamma].$$

Now we can deduce the desired estimate by using the Lemmas 2 and 3. Putting

$$w = \frac{\gamma^{1/\rho} + \beta^{1/\rho} - 2(\xi^*)^{1/\rho}}{\gamma^{1/\rho} - \beta^{1/\rho}} \qquad \text{and} \qquad r = w - (w^2 - 1)^{1/2}$$

we obtain

$$|P(\xi^{1/\rho})| \leqslant 2e^{-\beta n} r^{-n} T_n(\gamma^{1/\rho} + \beta^{1/\rho} - 1 - r(\gamma^{1/\rho} - \beta^{1/\rho})). \tag{16}$$

The right-hand side increases with decreasing values of r. Since $\xi^* \in [0, 1]$, we obviously have

$$r = \frac{1}{w + (w^2 - 1)^{1/2}} > \frac{1}{2w} > \frac{1}{2}\,\frac{\gamma^{1/\rho} - \beta^{1/\rho}}{\gamma^{1/\rho} + \beta^{1/\rho}}.$$

Replacing r in (16) by this lower bound and estimating $T_n(\cdot)$ with the help of Lemma 4, we get

$$|P(\xi^{1/\rho})| < \left\{2e^{-\beta}\,\frac{2(\gamma^{1/\rho} + \beta^{1/\rho})^2 - (\gamma^{1/\rho} - \beta^{1/\rho})^2}{\gamma^{1/\rho} - \beta^{1/\rho}}\right\}^n = \left\{e^{-\beta}\beta^{1/\rho} a\left(\frac{\gamma}{\beta}\right)\right\}^n, \tag{17}$$

where

$$a(h) = 2\,\frac{h^{2/\rho} + 6h^{1/\rho} + 1}{h^{1/\rho} - 1}.$$

Now set $\gamma = \beta(2(2^{1/2}) + 1)^\rho$. On comparing (15) and (17) we see that if

$$q_n = \left(1 - \exp\left(-\left(\beta - \frac{\tau + \epsilon}{\omega - \epsilon}\right) n\right)\right)^{1/n}$$

then we must have

$$\exp\left(-\frac{\tau + \epsilon}{\omega - \epsilon}\right) q_n < e^{-\beta}\beta^{1/\rho} a\left(\frac{\gamma}{\beta}\right) = e^{-\beta}\beta^{1/\rho}(16 + 8(2^{1/2})),$$

which is equivalent to

$$\beta < \frac{1}{\rho}\log\beta - \log q_n + \frac{\tau + \epsilon}{\omega - \epsilon} + \log(16 + 8(2^{1/2})). \qquad (18)$$

But since $\log q_n$ tends to zero as $n \to \infty$, the inequality

$$\frac{1}{\rho}\log\beta - \log q_n < \frac{1}{\rho}(\beta^{1/2}) = \frac{1}{\rho^2}\left\{1 + 2\rho^2\left(\frac{\tau + \epsilon}{\omega - \epsilon} + \log(16 + 8(2^{1/2})\right)\right\}^{1/2}$$
$$< \frac{1}{\rho^2} + \frac{\tau + \epsilon}{\omega - \epsilon} + \log(16 + 8(2^{1/2}))$$

holds for all sufficiently large n. Hence the right-hand side of (18) becomes smaller than β, a contradiction!

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